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Upper and Lower Solutions and Multiplicity Results

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We consider the second-order nonlinear differential equation with an unbounded right-hand side and two-point or nonlinear boundary conditions on a compact interval. Using the method of a priori estimates and the relation between the topological degree of the operators associated to the above boundary value problems and the strict lower and upper solutions, we get the multiplicity results for the problems. © 2000 Academic Press

1. INTRODUCTION

In [11] we studied the boundary value problems for the second-order differential equation

$$x'' = f(t, x, x'), \tag{1}$$

with f continuous and bounded on $J \times \mathbf{R}^2$, $J = [a, b] \subset \mathbf{R}$, i.e.,

$$\exists M \in (0, \infty) : |f(t, x, y)| < M \qquad \text{for all } (t, x, y) \in J \times \mathbf{R}^2. \tag{2}$$

We have considered the periodic conditions

$$x(a) = x(b), \qquad x'(a) = x'(b), \tag{3}$$

the Neumann conditions

$$x'(a) = 0, \qquad x'(b) = 0, \tag{4}$$

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and the nonlinear conditions

$$g_1(x(a), x'(a)) = 0, \qquad g_2(x(b), x'(b)) = 0, \tag{5}$$

where $g_1, g_2 \in C(\mathbf{R}^2)$ are increasing in the second argument and g_1 is nonincreasing and g_2 is nondecreasing in the first argument.

For the problems (1), (k), $k \in \{3, 4\}$, we have defined the associated operators L and N:

$$L\colon \operatorname{dom} L \to C(J),\ x \mapsto x'',\ \operatorname{dom} L = \{x \in C^2(J) : x \text{ fulfills } (k)\},$$

$$N\colon C^1(J) \to C(J),\ x \mapsto -f(\cdot, x(\cdot), x'(\cdot)),$$

and for the problem (1), (5) we have defined

$$L\colon C^2(J) \to C(J) \times \mathbf{R}^2,\ x \mapsto (x'', 0, 0),$$

$$N\colon C^1(J) \to C(J) \times \mathbf{R}^2,$$

$$x \mapsto (-f(\cdot, x(\cdot), x'(\cdot)), g_1(x(a), x'(a)), g_2(x(b), x'(b))).$$

The problems (1), (k), $k \in \{3, 4, 5\}$, can be written in the form of the operator equation

$$(L + N)x = 0. \tag{6}$$

For more details see [4, 8, 10]. In [11] we have proved for the degree d_L of the operator $L + N$ the following results:

THEOREM 1.1. *Suppose* $k \in \{3, 4, 5\}$. *Let* (2) *be fulfilled*, (6) *be the operator equation corresponding to the problem* (1), (k), *and let* σ_1, σ_2 *be strict lower and upper solutions of* (1), (k) *with*

$$\sigma_1(t) < \sigma_2(t) \qquad \text{for all } t \in J.$$

Then

$$d_L(L + N, \Omega_1) = 1,$$

where

$$\Omega_1 = \{x \in C^1(J) : \sigma_1(t) < x < \sigma_2(t),\ |x'(t)| < c \text{ for all } t \in J\}$$

with $c \geq (2M + r + 1)(b - a)$ *for* $k \in \{3, 4\}$ *and* $c \geq (2M + r + 1)(b - a) + 2(r + 1)/(b - a)$ *for* $k = 5$, $r = \|\sigma_1\|_{\max} + \|\sigma_2\|_{\max}$.

Theorem 1.1 concerns the case of well-ordered σ_1, σ_2. The case of σ_1, σ_2 ordered in the opposite way is described in Theorem 1.2.

THEOREM 1.2. *Suppose* $k \in \{3, 4, 5\}$. *Let* (2) *be fulfilled*; (6) *be the operator equation corresponding to the problem* (1), (k); *and* σ_1, σ_2 *be strict* *lower and upper solutions of* (1), (k) *satisfying*

$$\sigma_2(t) < \sigma_1(t) \qquad \text{for all } t \in J.$$

Then

$$d_L(L + N, \Omega_2) = -1,$$

where

$$\Omega_2 = \{x \in C^1(J) : \|x\|_{\max} < A, \|x'\|_{\max} < B, \exists t_x \in J : \sigma_2(t_x) < x(t_x) < \sigma_1(t_x)\},$$

with $B \geq 2(b-a)M$, $A \geq \|\sigma_1\|_{\max} + \|\sigma_2\|_{\max} + 2(b-a)^2 M$ *for* $k \in \{3, 4\}$, *and* $B \geq 2(b-a)M + \|\sigma_2'\|_{\max}$, $A \geq \|\sigma_1\|_{\max} + \|\sigma_2\|_{\max} + (b-a)B$ *for* $k = 5$.

Here, we extend the results of Theorems 1.1 and 1.2 onto differential equations with an unbounded right-hand side f. For the whole paper we suppose that $k \in \{3, 4, 5\}$; that (6) is the operator equation corresponding to the problem (1), (k); and that σ_1, σ_2 are strict lower and upper solutions of (1), (k).

In the mathematical literature there are many existence results via the lower and upper solutions method. We should mention the papers by H. B. Thompson [14, 15], where the existence of solutions to the equation (1) with fully nonlinear two-point boundary conditions has been established and many references and further information can be found. The boundary conditions (3), (4), and (5) in our paper are a specification of those in [14, 15]. However, thanks to their special properties we need only one-sided growth restrictions for f instead of the two-sided ones which are required in [14, 15]. Moreover, the crucial assumption in [14, 15] is the existence of a lower solution σ_1 and an upper solution σ_2 which are well ordered, i.e., $\sigma_1 \leq \sigma_2$. But we consider also the opposite order $\sigma_2 < \sigma_1$. Our theorems on multiplicity results from Section 4 combine both types of ordering of lower and upper solutions and so they cannot be obtained by the approach presented in [14, 15]. For other results concerning nonordered or reverse-ordered lower and upper solutions we refer the reader to the papers [2, 5, 6, 9]. In the literature we can find multiplicity results reached via proper modifications of the Leggett–Williams fixed point theorem. In this way the existence of three positive solutions of the autonomous differential equation with the Dirichlet boundary conditions has been proved in [1]. It should be interesting to apply this method to the nonautonomous differential equation (1) with the boundary condition (3), (4), or (5) and to compare the results obtained with our results in Section 4.

2. NAGUMO–KNOBLOCH–SCHMITT CONDITIONS

Using the method of a priori estimates, we can replace the condition regarding (2) in Theorem 1.1 with the Nagumo–Knobloch–Schmitt condition with bounding functions φ_1, φ_2,

$$\exists \varphi_1, \varphi_2 \in C^1(K) : \varphi_1(t, \sigma_i(t)) \le \sigma_i'(t), \varphi_2(t, \sigma_i(t)) \ge \sigma_i'(t),$$

$$f(t, x, \varphi_1(t, x) < \frac{\partial \varphi_1(t, x)}{\partial t} + \frac{\partial \varphi_1(t, x)}{\partial x} \varphi_1(t, x),$$

$$f(t, x, \varphi_2(t, x) > \frac{\partial \varphi_2(t, x)}{\partial t} + \frac{\partial \varphi_2(t, x)}{\partial x} \varphi_2(t, x),$$

for $i \in \{1, 2\}$ and for all $(t, x) \in K = J \times [\sigma_1(t), \sigma_2(t)]$.

THEOREM 2.1. *Let* (7) *be fulfilled and let*

$$\sigma_1(t) < \sigma_2(t) \qquad \textit{for all } t \in J.$$

Further, suppose that for $k = 3$,

$$(\varphi_i(b, x) - \varphi_i(a, x))(-1)^i \ge 0; \tag{8}$$

for $k = 4$,

$$(\varphi_i(b, x) - \sigma_i'(b))(-1)^i > 0; \tag{9}$$

and for $k = 5$,

$$g_2(x, \varphi_i(b, x))(-1)^i > 0, \tag{10}$$

with $i = 1, 2,\ x \in [\sigma_1(t), \sigma_2(t)]$.
Then

$$d_L(L + N, \Omega_3) = 1,$$

where

$$\Omega_3 = \{x \in C^1(J) : \sigma_1(t) < x(t) < \sigma_2(t),$$
$$\varphi_1(t, x) < x'(t) < \varphi_2(t, x) \textit{ on } K\}.$$

Proof. Put

$$\sigma(t, x) = \begin{cases} \sigma_2(t) & \text{for } x > \sigma_2(t) \\ x & \text{for } \sigma_1(t) \le x \le \sigma_2(t) \\ \sigma_1(t) & \text{for } x < \sigma_1(t), \end{cases}$$

$$\varphi(t,x,y) = \begin{cases} \varphi_2(t,x) & \text{for } y > \varphi_2(t,x) \\ y & \text{for } \varphi_1(t,x) \le y \le \varphi_2(t,x) \\ \varphi_1(t,x) & \text{for } y < \varphi_1(t,x), \end{cases}$$

$$f^*(t,x,y) = f(t,\sigma(t,x),\varphi(t,x,y)),$$

and consider the auxiliary equation

$$x'' = f^*(t,x,x'). \tag{11}$$

We can see that f^* satisfies (2) with $M > \max\{|f(t,x,y)| : (t,x,y) \in J \times [\sigma_1(t), \sigma_2(t)] \times [\varphi_1(t,x), \varphi_2(t,x)]\}$ and σ_1, σ_2 being the strict lower and upper solutions for the problem (11), (k), $k \in \{3,4,5\}$. Let us define the set

$$\Omega = \{x \in C^1(J) : \sigma_1(t) < x < \sigma_2(t), |x'(t)| < c \text{ for all } t \in J\},$$

where

$$c \ge (2M + r + 1)(b - a) \qquad \text{for } k \in \{3,4\},$$
$$c \ge (2M + r + 1)(b - a) + 2(r + 1)/(b - a) \qquad \text{for } k = 5,$$

and

$$r = \|\sigma_1\|_{\max} + \|\sigma_2\|_{\max}.$$

Further, for $k \in \{3,4\}$ we put

$$N^* : C^1(J) \to C(J),\ x \mapsto -f^*(\cdot, x(\cdot), x'(\cdot)),$$

and for $k = 5$ we put

$$N^* : C^1(J) \to C(J) \times \mathbf{R}^2,$$
$$x \mapsto (-f^*(\cdot, x(\cdot), x'(\cdot)), g_1(x(a), x'(a)), g_2(x(b), x'(b))).$$

Then, by Theorem 1.1,

$$d_L(L + N^*, \Omega) = 1. \tag{12}$$

Let us prove that for any solution u of the problem (11), (k), $k \in \{3,4,5\}$, the implication $u \in \Omega \Rightarrow u \in \Omega_3$ holds. Put $z_2(t) = u'(t) - \varphi_2(t,u)$ and $z_1(t) = \varphi_1(t,u) - u'(t)$ and suppose that there exists $i \in \{1,2\}$ and $t_0 \in J$ such that

$$\max\{z_i(t) : t \in J\} = z_i(t_0) \ge 0.$$

Then $z_i'(t_0) \le 0$ for $t_0 \in [a, b)$. On the other hand, by (7), $z_i'(t_0) > 0$, a contradiction. Now, suppose that $t_0 = b$. For $k = 3$, we have $u'(b) = u'(a)$ and, by (8), $(\varphi_i(b, u(b)) - \varphi_i(a, u(a)))(-1)^i \ge 0$, thus $z_2(b) = u'(b) - \varphi_2(b, u(b)) \le u'(a) - \varphi_2(a, u(a)) = z_2(a)$, and $z_1(b) = \varphi_1(b, u(b)) - u'(b) \le \varphi_1(a, u(a)) - u'(a) \le z_1(a)$. So $z_i(b) = z_i(a)$ and we can put $t_0 = a$. For $k = 4$ we have $u'(a) = u'(b) = 0$ and $\sigma_1'(a) \ge 0$, $\sigma_1'(b) \le 0$, $\sigma_2'(a) \le 0$, $\sigma_2'(b) \ge 0$, thus, by (9), $\varphi_1(b, u(b)) < \sigma_1'(b) \le 0$, $\varphi_2(b, u(b)) > \sigma_2'(b) \ge 0$, and $z_2(b) < 0$, $z_1(b) < 0$, which is a contradiction. For $k = 5$, according to (10), we have $g_2(u(b), u'(b)) \ge g_2(u(b), \varphi_2(b, u(b)) > 0$ if $i = 2$, and $g_2(u(b), u'(b)) \le g_2(u(b), \varphi_1(b, u(b)) < 0$ if $i = 1$. In both cases we get a contradiction. Therefore

$$\varphi_1(t, u(t)) < u'(t) < \varphi_2(t, u(t)) \qquad \text{on } K$$

and thus $u \in \Omega_3$. By the excision property of the degree we get from (12)

$$d_L(L + N^*, \Omega_3) = 1,$$

and since $N = N^*$ on Ω_3 Theorem 2.1 is proved.

For the constant functions $\sigma_1, \sigma_2, \varphi_1, \varphi_2$ Theorem 2.1 implies

COROLLARY 2.2. *Suppose that there exist real numbers $r_1 < r_2$, $c_1 < 0 < c_2$, such that*

$$f(t, r_1, 0) < 0, \qquad f(t, r_2, 0) > 0, \tag{13}$$

$$f(t, x, c_1) < 0, \qquad f(t, x, c_2) > 0, \tag{14}$$

for all $(t, x) \in J \times [r_1, r_2]$.

If $k = 5$ we suppose moreover that for $x \in [r_1, r_2]$,

$$g_2(x, c_i)(-1)^i > 0, \qquad i = 1, 2, \tag{15}$$

$$g_1(r_1, 0) \ge 0, \qquad g_1(r_2, 0) \le 0, \tag{16}$$

$$g_2(r_1, 0) \le 0, \qquad g_2(r_2, 0) \ge 0.$$

Then

$$d_L(L + N, \Omega_4) = 1,$$

where

$$\Omega_4 = \{x \in C^1(J) : r_1 < x(t) < r_2, c_1 < x'(t) < c_2, \forall t \in J\}.$$

Now, let us consider the special case of bounding functions depending on t only.

$$\exists \beta_1, \beta_2 \in C^1(J)\colon \beta_1(t) \leq \sigma_i'(t),\ \beta_2(t) \geq \sigma_i'(t),$$
$$f(t, x, \beta_1(t)) < \beta_1'(t), \qquad f(t, x, \beta_2(t)) > \beta_2'(t), \tag{17}$$

for all $(t, x) \in J \times [s_2, s_1]$, where $s_2 = \min\{\sigma_2(t)\colon t \in J\} - \int_a^b \gamma(t)\, dt$, $s_1 = \max\{\sigma_1(t)\colon t \in J\} + \int_a^b \gamma(t)\, dt$, $\gamma(t) = \max\{|\beta_1(t)|, |\beta_2(t)|\}$.

THEOREM 2.3. *Let* (17) *be fulfilled and let*

$$\sigma_2(t) < \sigma_1(t) \qquad \text{for all } t \in J.$$

Further suppose that for $k = 3$

$$(\beta_i(b) - \beta_i(a))(-1)^i \geq 0, \tag{18}$$

for $k = 4$

$$(\beta_i(b) - \sigma_i'(b))(-1)^i > 0, \tag{19}$$

and for $k = 5$

$$g_2(x, \beta_i(b))(-1)^i > 0, \tag{20}$$

with $i \in \{1, 2\}$, $x \in [s_2, s_1]$.

Then

$$d_L(L + N, \Omega_5) = -1,$$

where

$$\Omega_5 = \Big\{x \in C^1(J) : s_2 < x(t) < s_1,\ \beta_1(t) < x'(t) < \beta_2(t) \text{ for all } t \in J,$$
$$\exists t_x \in J\colon \sigma_2(t_x) < x(t_x) < \sigma_1(t_x)\Big\}.$$

Proof. Put

$$\rho(x) = \begin{cases} s_1 & \text{for } x > s_1 \\ x & \text{for } s_2 \leq x \leq s_1 \\ s_2 & \text{for } x < s_2, \end{cases}$$

$$\beta(t, y) = \begin{cases} \beta_2(t) & \text{for } y > \beta_2(t) \\ y & \text{for } \beta_1(t) \leq y \leq \beta_2(t) \\ \beta_1(t) & \text{for } y < \beta_1(t), \end{cases}$$

$$f^*(t, x, y) = f(t, \rho(x), \beta(t, y)),$$

and consider the equation (11). We can see that f^* satisfies (2) with $M > \max\{|f(t,x,y)| : (t,x,y) \in J \times [s_2, s_1] \times [\beta_1(t), \beta_2(t)]\}$ and σ_1, σ_2 are strict lower and upper solutions for the problem (11), (k), $k \in \{3,4,5\}$. Let us define the set

$$\Omega = \big\{x \in C^1(J) : \|x\|_{\max} < A, \|x'\|_{\max} < B,$$
$$\exists t_x \in J \colon \sigma_2(t_x) < x(t_x) < \sigma_1(t_x)\big\}$$

with $B = 2(b-a)M + \|\gamma\|_{\max}$, $A = \|\sigma_1\|_{\max} + \|\sigma_2\|_{\max} + (b-a)B$ for $k = 3, 4$; and $B = 2(b-a)M + \|\gamma\|_{\max} + \|\sigma_2'\|_{\max}$, $A = \|\sigma_1\|_{\max} + \|\sigma_2\|_{\max} + (b-a)B$ for $k = 5$.

Further, we define the operator N^* as in the proof of Theorem 2.1, and using Theorem 1.2 we get

$$d_L(L + N^*, \Omega) = -1.$$

We can follow the proof of Theorem 2.1, and using (17)–(20) we prove that for any solution u of (11), (k), $k \in \{3,4,5\}$,

$$u \in \Omega \Rightarrow \beta_1(t) < u'(t) < \beta_2(t), \qquad \text{for all } t \in J.$$

Integrating the last inequality we get

$$s_2 < u(t) < s_1 \qquad \text{for all } t \in J,$$

i.e., $u \in \Omega_5$. By the excision property of the degree we get

$$d_L(L + N^*, \Omega_5) = -1,$$

and since $N = N^*$ on Ω_5 Theorem 2.3 is proved.

COROLLARY 2.4. *Suppose that there exist real numbers* $r_1 > r_2$, $c_1 < 0 < c_2$, *such that* (13) *and* (14) *are satisfied for all* $(t,x) \in J \times [r_2 + c_1(b-a), r_1 + c_2(b-a)]$. *If* $k = 5$, *we suppose that* (15), (16) *are satisfied for* $x \in [r_2 + c_1(b-a), r_1 + c_2(b-a)]$.

Then

$$d_L(L + N, \Omega_6) = -1,$$

where

$$\Omega_6 = \big\{x \in C^1(J) : r_2 + c_1(b-a) < x(t) < r_1 + c_2(b-a),$$
$$\times c_1 < x'(t) < c_2, \forall t \in J, \exists t_x \in J \colon r_2 < x(t_x) < r_1\big\}.$$

EXAMPLE. Suppose $f_1, f_2, f_3 \in C(J)$, $k, m \in \mathbf{N}$. The function

$$f(t,x,y) = f_1(t)x^{2k+1} + f_2(t)y^{2m+1} + f_3(t)$$

satisfies the conditions of Corollary 2.2 if $f_1, f_2 > 0$ on J; and it satisfies the conditions of Corollary 2.4 if $f_1 < 0$, $f_2 > 0$ on J and either $m > k$ or $m = k$, $f_2(t) > \|f_1\|_{\max}(b-a)^{2k+1}$ for all $t \in J$.

3. ONE-SIDED GROWTH CONDITIONS

Other types of conditions which can be used instead of (2) in Theorems 1.1 and 1.2 are the one-sided growth conditions which were used by Kiguradze [7] in some existence theorems.

1. THE ONE-SIDED BERNSTEIN–NAGUMO CONDITION. $\exists \omega \in C(\mathbf{R}_+)$, ω positive, $\int_0^\infty (ds/\omega(s)) = \infty$, and

$$f(t,x,y) \le \omega(|y|)\cdot(1+|y|)\ \forall (t,x) \in J \times [\sigma_1(t), \sigma_2(t)] \times \mathbf{R}. \quad (21)$$

2. THE ONE-SIDED LINEAR GROWTH CONDITION. $\exists a_1, a_2 \in (0,\infty)$, $\rho \in C(J \times \mathbf{R})$, non-negative and non-decreasing, in the second argument, such that

$$f(t,x,y) \le a_1|x| + a_2|y| + \rho(t, |x|+|y|) \qquad \forall (t,x,y) \in J \times \mathbf{R}^2, \quad (22)$$

where

$$a_1(b-a)^2 + a_2(b-a) < 1 \quad (23)$$

and

$$\lim_{z\to\infty} \frac{1}{z}\int_a^b \rho(t,z)\,dt = 0.$$

Note. Let us remember that if f satisfies (22) it satisfies (21) as well.

First, we will prove the lemmas on a priori estimates for solutions of the problems (1), (k), $k \in \{3,4,5\}$.

LEMMA 3.1. *Suppose*

$$\sigma_1(t) < \sigma_2(t) \qquad \textit{for all } t \in J.$$

Let (21) *be satisfied. If* $k = 5$, *suppose moreover*

$$\lim_{y\to\infty} g_1(r_2, y) > 0, \qquad \lim_{y\to-\infty} g_2(r_2, y) < 0, \quad (24)$$

$r_1 = \min\{\sigma_1(t) : t \in J\}$, $r_2 = \max\{\sigma_2(t) : t \in J\}$.

Then there exists $\mu^* \in (0, \infty)$ *such that for any solution* u *of the problem* (1), (k), *the implication*

$$\sigma_1(t) < u(t) < \sigma_2(t) \qquad on\ J \Rightarrow \|u'\|_{\max} < \mu^* \tag{25}$$

is valid.

Proof. Let u be a solution of (1), (k) and let

$$\sigma_1(t) < u(t) < \sigma_2(t) \qquad \text{for all } t \in J. \tag{26}$$

Let us put $r = \|\sigma_1\|_{\max} + \|\sigma_2\|_{\max}$, $\mu = \max\{|u'(t)|: t \in J\}$. The condition (21) implies that

$$u'' \le \omega(|u'(t)|)(1 + |u'(t)|), \qquad \forall t \in J. \tag{27}$$

1. Let $k = 3$. Then we can find $t_0 \in (a, b)$ such that $u'(t_0) = 0$. From (21) it follows that there exist $\mu_1, \mu^* \in (1, \infty)$, $\mu_1 < \mu^*$, such that

$$\int_1^{\mu_1} \frac{ds}{\omega(s)} = K > 2r, \qquad \int_1^{\mu^*} \frac{ds}{\omega(s)} > K + 2r. \tag{28}$$

(a) Suppose that there exists $t_1 \in (t_0, b]$ such that

$$\max\{u'(t): t \in [t_0, b]\} = u'(t_1) = c_1 > 1.$$

Then we can find $\alpha_1 \in (t_0, t_1)$ such that

$$u'(\alpha_1) = 1 \qquad \text{and} \qquad u'(t) > 1\ \forall t \in (\alpha_1, t_1].$$

Integrating (27) from α_1 to t_1 we get

$$\int_{\alpha_1}^{t_1} \frac{u''(t)\, dt}{\omega(u'(t))} \le 2\int_{\alpha_1}^{t_1} u'(t)\, dt,$$

thus

$$\int_1^{c_1} \frac{ds}{\omega(s)} \le 2r,$$

which gives $c_1 < \mu_1$. Therefore $u'(t) < \mu_1$ for all $t \in [t_0, b]$, $u'(a) < \mu_1$. Further suppose that there exists $t_2 \in [a, t_0)$ such that

$$\max\{u'(t): t \in [a, t_0]\} = u'(t_2) = c_2 > \mu_1.$$

Then we can find $\alpha_2 \in [a, t_2)$ such that

$$u'(\alpha_2) = \mu_1 \qquad \text{and} \qquad u'(t) > \mu_1 \text{ for all } t \in (\alpha_2, t_2].$$

Integrating (27) from α_2 to t_2 we get

$$\int_1^{c_2} \frac{ds}{\omega(s)} \le K + 2r,$$

which gives $c_2 < \mu^*$. Thus we have proven

$$u'(t) < \mu^* \qquad \text{for all } t \in J. \tag{29}$$

(b) Now, we will estimate u' from below. Suppose that there exists $t_3 \in [a, t_0)$ such that

$$\min\{u'(t): t \in [a, t_0]\} = u'(t_3) = -c_3 < -1.$$

If we put $v'(t) = -u'(t)$, we can prove as in (a) that $c_3 < \mu_1$, i.e. $u'(t) > -\mu_1$ on $[a, t_0]$, $u'(b) > -\mu_1$. Supposing that

$$\min\{u'(t): t \in [t_0, b]\} = u'(t_4) = -c_4 < -\mu_1,$$

we can also use for $v' = -u'$ the same argument as in (a) and get $c_4 < \mu^*$, i.e.,

$$-\mu^* < u'(t) \qquad \text{for all } t \in J. \tag{30}$$

2. Let $k = 4$. From (21) it follows that there exists $\mu^* \in (1, \infty)$ such that

$$\int_1^{\mu^*} \frac{ds}{\omega(s)} > 2r. \tag{31}$$

Now, we can use the same considerations as for the periodic problem (1), (3), but instead of proving the estimate on $[t_0, b]$ and then on $[a, t_0]$ in step (a), and on $[a, t_0]$ and then on $[t_0, b]$ in step (b), we can put $t_0 = a$ in step (a) and get (29) and then put $t_0 = b$ in step (b) and get (30).

3. Let $k = 5$. Then (24) guarantees the existence of $\gamma \in (1, \infty)$ such that for any solution x of (1), (5) satisfying (26),

$$x'(a) < \gamma, \qquad x'(b) > -\gamma. \tag{32}$$

Otherwise, we could find a sequence of solutions $\{x_n\}_1^\infty$ of (1), (5) satisfying (26) with $x_n'(a) \to \infty$ or $x_n'(b) \to -\infty$ for $n \to \infty$. So, there exists $n_0 \in \mathbf{N}$ such that $g_1(x_{n_0}(a), x'_{n_0}(a)) > 0$ or $g_2(x_{n_0}(b), x'_{n_0}(b)) < 0$, a contradiction.

Further, from (21) it follows that there exists $\mu^* \in (\gamma, \infty)$ such that

$$\int_1^{\mu^*} \frac{ds}{\omega(s)} > 2r + \int_1^{\gamma} \frac{ds}{\omega(s)}. \tag{33}$$

Then we can argue as for $k = 4$, and using (33) we get (29) in step (a), and (30) in step (b).

LEMMA 3.2. *Let $r_1, r_2 \in \mathbf{R}$, $r_1 < r_2$, and let* (22) *be satisfied. If $k = 5$, suppose moreover that*

$$\lim_{y \to \infty} g_1(x, y) > 0, \qquad \lim_{y \to -\infty} g_2(x, y) < 0, \tag{34}$$

uniformly for $x \in \mathbf{R}_+$.

Then there exists $\nu^* \in (0, \infty)$ such that for any solution u of the problem $(1), (k)$, *the implication*

$$\exists t_u \in J : r_1 < u(t_u) < r_2 \Rightarrow \|u'\|_{\max} < \nu^* \tag{35}$$

is valid.

Proof. Let x be a solution of $(1), (k)$, $k \in \{3, 4, 5\}$, and let there exist a $t_x \in J$ such that $r_1 < u(t_x) < r_2$. Let us put $r = |r_1| + |r_2|$, $\mu = \max\{|x'(t)|: t \in J\}$. The condition (22) implies that

$$x''(t) \le a_1|x(t)| + a_2|x'(t)| + \rho(t, |x| + |x'|) \qquad \forall t \in J. \tag{36}$$

1. Let $k = 3$. We have $|x(t)| \le \mu(b - a) + r$ for all $t \in J$ and we can find $t_0 \in (a, b)$ such that $x'(t_0) = 0$.

(a) Integrating (36) from t_0 to $t \in (t_0, b]$, we get

$$x'(t) \le A(\mu, t_0, b)\ \forall t \in [t_0, b] \qquad \text{and} \qquad x'(a) \le A(\mu, t_0, b), \tag{37}$$

where

$$A(\mu, t_0, b) = a_1[\mu(b - a) + r](b - t_0) + a_2\mu(b - t_0) + \int_{t_0}^{b} \rho(s, \mu(b - a + 1) + r)\, ds.$$

Integrating (36) from a to $t \in (a, t_0]$ and using (37), we get

$$x'(t) \le A(\mu, t_0, b) + A(\mu, a, t_0) = A(\mu, a, b) \qquad \forall t \in [a, t_0].$$

Thus

$$x'(t) \le A(\mu, a, b) \qquad \text{for all } t \in J. \tag{38}$$

(b) Now, let us integrate (36) from $t \in [a, t_0)$ to t_0:

$$-x'(t) \le A(\mu, a, t_0) \ \forall t \in [a, t_0] \qquad \text{and} \qquad -x'(b) \le A(\mu, a.t_0). \tag{39}$$

Finally, using (39) and integrating (36) from $t \in [t_0, b)$ to b, we have

$$-x'(t) \le A(\mu, a, t_0) + A(\mu, t_0, b) = A(\mu, a, b) \qquad \forall t \in \{t_0, b].$$

Therefore

$$-x'(t) \le A(\mu, a, b) \qquad \text{for all } t \in J. \tag{40}$$

Inequalities (38) and (40) give

$$\mu \le A(\mu, a, b). \tag{41}$$

2. Let $k = 4$. Then we can put $t_0 = a$ in step (a) and $t_0 = b$ in step (b) and get (41).

3. Let $k = 5$. We can show as in the proof of Lemma 3.1, Part 3, that (32) is valid for any solution x of (1), (5). For proving the estimation of the first derivatives of the solutions we can argue as for $k = 4$ and get

$$\mu \le \gamma + A(\mu, a, b). \tag{42}$$

Let us show that there exists a $\nu^* \in (0, \infty)$ such that

$$\max\{|x'(t)|: t \in J\} = \mu < \nu^*$$

for any solution x of the problem (1), (k), $k \in \{3, 4, 5\}$. Suppose that such a constant ν^* does not exist. Then we can find a sequence of solutions $\{x_n\}_1^\infty$ of the problem (1), (k) and the associated sequence of $\{\mu_n\}_1^\infty$ such that $\lim_{n\to\infty} \mu_n = \infty$. If $k \in \{3, 4\}$ we get, according to (41), $\mu_n \le A(\mu_n, a, b)$, i.e.,

$$1 \le \frac{1}{\mu_n} A(\mu_n, a, b) = a_1(b-a)^2 + a_2(b-a)$$
$$+ \frac{1}{\mu_n} a_1 r(b-a) + \frac{1}{\mu_n} \int_a^b \rho(s, \mu_n(1 + b - a) + r)\, ds$$

and if $k = 5$ we get from (42)

$$1 \le \frac{1}{\mu_n}\gamma + \frac{1}{\mu_n} A(\mu_n, a, b).$$

Provided $\mu_n \to \infty$ we get for $k = \{3, 4, 5\}$

$$1 \le a_1(b-a)^2 + a_2(b-a),$$

which contradicts (23). So the implication (35) is valid and Lemma 3.2 is proved.

THEOREM 3.3. *Let* (21) *be fulfilled and let*

$$\sigma_1(t) < \sigma_2(t) \qquad \textit{for all } t \in J.$$

If $k = 5$ *suppose moreover* (24).

Then there exists $r^* \in (0, \infty)$ *such that*

$$d_L(L + N, \Omega_6) = 1,$$

where

$$\Omega_6 = \left\{x \in C^1(J) \colon \sigma_1(t) < x(t) < \sigma_2(t)\ \forall t \in J, \|x'\|_{\max} < r^*\right\}.$$

Proof. Let μ^* be the constant from Lemma 3.1. Put

$$r = \|\sigma_1\|_{\max} + \|\sigma_2\|_{\max}, \qquad r^* = r + \max\{\mu^*, \|\sigma_1'\|_{\max}, \|\sigma_2'\|_{\max}\},$$

$$\chi(s, \phi) = \begin{cases} 1 & \text{for } 0 \le s \le \phi \\ 2 - s/\phi & \text{for } \phi < s < 2\phi, \\ 0 & \text{for } s \ge 2\phi \end{cases}$$

$$f^*(t, x, y) = \chi(|x| + |y|, r^*) f(t, x, y),$$

and define for $k \in \{3, 4\}$

$$N^* \colon C^1(J) \to C(J),\ x \mapsto -f^*(\cdot, x(\cdot), x'(\cdot)),$$

and for $k = 5$

$$N^* \colon C^1(J) \to C(J) \times \mathbf{R}^2,$$

$$x \mapsto \left(-f^*(\cdot, x(\cdot), x'(\cdot)), g_1(x(a), x'(a)), g_2(x(b), x'(b))\right).$$

The differential equation

$$x'' = f^*(t, x, x') \tag{43}$$

has also σ_1, σ_2 as its strict lower and upper solutions and the function f^* satisfies (2) with $M = 1 + \max\{|f^*(t, x, y)| \colon t \in J, |x| + |y| \le 2r^*\}$. Therefore, by Theorem 1.1,

$$d_L(L + N^*, \Omega) = 1, \tag{44}$$

where

$$\Omega = \{x \in C^1(J): \sigma_1(t) < x(t) < \sigma_2(t), |x'(t)| < c \ \forall t \in J\}$$

with

$$c = (2M + r + 1)(b - a) + \frac{2(r+1)}{b - a} + r^*.$$

We can see that f^* fulfills (21) and thus, by Lemma 3.1, the implication (25) is valid. This means that any solution $u \in \Omega$ of the problem (43), (k) belongs to Ω_6. Thus, by the excision property of the degree

$$d_L(L + N^*, \Omega_6) = 1,$$

and since $N^* = N$ on Ω_6 Theorem 3.3 is proved.

THEOREM 3.4. *Let* (22) *be fulfilled and let*

$$\sigma_2(t) < \sigma_1(t) \ \textit{for all } t \in J.$$

If $k = 5$, *suppose moreover* (34).

Then there exists $r^* \in (0, \infty)$ *such that*

$$d_L(L + N, \Omega_7) = -1,$$

where

$$\Omega_7 = \{x \in C^1(J) : \|x\|_{\max} + \|x'\|_{\max} < r^*, \\ \exists t_x \in J : \sigma_2(t_x) < x(t_x) < \sigma_1(t_x)\}.$$

Proof. Let ν^* be the constant from Lemma 3.2. Put $r_1 = \min\{\sigma_2(t): t \in J\}$, $r_2 = \max\{\sigma_1(t): t \in J\}$, $r^* = \nu^*(1 + b - a) + |r_1| + |r_2| + \|\sigma_1'\|_{\max} + \|\sigma_2'\|_{\max}$. Now, we can follow the proof of Theorem 3.3, define f^* and N^* in the same way, and using Theorem 1.2 we get (44), where

$$\Omega = \{x \in C^1(J) : \|x\|_{\max} < A, \|x'\| < B, \\ \exists t_x \in J: \sigma_2(t_x) < x(t_x) < \sigma_1(t_x)\},$$

with $B = 2(b - a)M + r^*$, $A = \|\sigma_1\|_{\max} + \|\sigma_2\|_{\max} + 2(b - a)^2 M$.

We can see that f^* fulfills (22) and thus, using Lemma 3.2, we can finish the proof similarly to that of Theorem 3.3.

4. MULTIPLICITY RESULTS

Let us suppose that σ_1, σ_2, and σ_3 are strict lower, upper, and lower solutions of (1), (k), $k \in \{3, 4, 5\}$. Using Theorems 2.1 and 2.3 we get the following multiplicity result:

THEOREM 4.1. *Suppose that*

$$\sigma_1(t) < \sigma_2(t) < \sigma_3(t) \qquad \textit{for all } t \in J \tag{45}$$

and that (17) *and, according to* k, *the condition* (18) *or* (19) *or* (20) *are fulfilled for all* $(t, x) \in J \times [\sigma_1(t), s_3]$, *where* $s_3 = \max\{\sigma_3(t)\colon\ t \in J\} + \int_a^b \gamma(t)\, dt$.

Then (1), (k) *has at least two different solutions* u, v *satisfying*

$$\sigma_1(t) < u(t) < \sigma_2(t), \qquad \sigma_1(t) < v(t) \textit{ for all } t \in J,$$

$$\sigma_2(t_v) < v(t_v) < \sigma_3(t_v) \qquad \textit{for a } t_v \in J. \tag{46}$$

Proof. Since (17)–(20) is the special case of (7)–(10), the existence of a solution u lying between σ_1 and σ_2 follows from Theorem 2.1. The existence of the second solution v satisfying (46) follows from Theorem 2.3. The inequality $\sigma_1 < v$ on J can be proved in the same way as for u in the proof of Theorem 2.1.

Similarly, by means of Theorem 3.3 and Theorem 3.4 and the fact that (22) and (34) are the special cases of (21) and (24), we get:

THEOREM 4.2. *Let us suppose that* (45) *and* (22) *are fulfilled, and for* $k = 5$ *suppose moreover* (34). *Then the assertion of Theorem* 4.1 *is valid.*

Now, let us consider the dual situation, where σ_3 is an upper solution of (1), (k).

THEOREM 4.3. *Suppose that*

$$\sigma_3(t) < \sigma_1(t) < \sigma_2(t) \qquad \textit{for all } t \in J, \tag{47}$$

and that (17) *and, according to* k, *the condition* (18) *or* (19) *or* (20) *are fulfilled for all* $(t, x) \in J \times [b_3, \sigma_2(t)]$, *where* $b_3 = \min\{\sigma_3(t)\colon\ t \in J\} - \int_a^b \gamma(t)\, dt$.

Then (1), (k) *has at least two different solutions* u, v *satisfying*

$$\sigma_1(t) < u(t) < \sigma_2(t), v(t) < \sigma_2(t) \qquad \textit{for all } t \in J,$$

$$\sigma_3(t_v) < v(t_v) < \sigma_1(t_v) \qquad \textit{for a } t_v \in J.$$

THEOREM 4.4. *Let us suppose that* (47) *and* (22) *are fulfilled, and for* $k = 5$ *suppose moreover* (34). *Then the assertion of Theorem* 4.3 *is valid.*

For constant lower and upper solutions we can generalize the theorems from [12] concerning the multiplicity results of the Ambrosetti–Prodi type for the periodic problem.

THEOREM 4.5. *Suppose* $k \in \{3, 4\}$. *Let* $n \in \mathbf{N}$, $n \geq 2$, $c_1, c_2, s_1, r_1, \dots, r_{n+1} \in \mathbf{R}$ *be such that*

$$r_1 < r_2 < \cdots < r_{n+1}, \tag{48}$$

$$c_1 < 0 < c_2,$$

$$(s_1 - f(t, r_i, 0))(-1)^i > 0 \qquad \text{for all } t \in J,\ i \in \{1, \dots, n\}, \tag{49}$$

and

$$f(t, x, c_1) < 0, \qquad f(t, x, c_2) > 0 \tag{50}$$

for all $(t, x) \in J \times [r_1, r^*]$, *where*

$$r^* = \begin{cases} r_{n+1} & \text{for } n \text{ odd} \\ r_{n+1} + \max\{|c_1|, c_2\}(b - a) & \text{for } n \text{ even}. \end{cases} \tag{51}$$

Then there exist $s_2, s_3 \in (-\infty, s_1)$, $s_3 \leq s_2$, *such that the problem*

$$x'' + f(t, x, x') = s, (k) \tag{52}$$

has:

(i) *at least n different solutions* u_i, $i = 1, \dots, n$, *satisfying*

$$r_1 < u_i(t) < r^* \qquad \text{for all } t \in J,\ i \in \{1, \dots, n\}; \tag{53}$$

(ii) *at least* $((n + 1)/2)(n/2)$ *solutions satisfying* (53) *for* $s = s_2$ *and n odd* (*even*);

(iii) *provided* $s_3 < s_2$ *at least one solution satisfying* (53) *for* $s \in [s_3, s_2)$;

(iv) *no solution satisfying* (53) *for* $s < s_3$.

Proof. Let $j \in \{1, \dots, n + 1\}$. The condition (49) implies that there exists $s_2 < s_1$ such that for j odd (even) r_j is a strict lower (upper) solution to (52) for $s \in (s_2, s_1]$. Therefore, by using Theorem 4.1 we get (i). For $s = s_2$ at least one of the strict upper solutions r_j of the problem (52) becomes nonstrict and so at least two solutions of this problem can be the same. In the case where all the upper solutions become nonstrict for $s = s_2$, all neighbor pairs of solutions of (52) can be identical. Thus (ii) is true. Suppose that x is a solution of (52) satisfying (53). Put $m < \min\{f(t, x, y): (t, x, y) \in J \times [r_1, r^*] \times [c_1, c_2]\}$. Then, by integrating the equation (52) from a to b we get $m < s$. Thus for $s \leq m$ the problem (52) has no solution satisfying (53). Suppose that for some $s^* \in (m, s_1)$ the problem (52) has a solution u^*. Then there exists a solution of (52) for all $s \in [s^*, s_1]$ because u^* is an upper solution, r_1 is a lower solution of (52) for $s \in [s^*, s_1]$, and $u^*(t) > r_1$ on J. So, we can set $s_3 = \inf\{s\colon s < s_1$, (52) has a solution satisfying (53)$\}$; then $s_3 \in (m, s_2]$. If $s_3 < s_2$, we consider a sequence $\{\sigma_n\} \subset (s_3, s_2)$ converging to s_3 and the corresponding sequence of solutions $\{u_n\}$ of the problems $\{(52), s = \sigma_n\}$. This sequence is equi-bounded and equi-continuous in $C^1(J)$, and by the Arzelà–Ascoli theorem we can choose a subsequence converging in the space $C^1(J)$ to a solution of (52) for $s = s_3$. Thus (iii) and (iv) are valid.

Similarly, we can prove:

THEOREM 4.6. *If we change the inequality* (49) *in Theorem* 4.5 *to the opposite one*, *i.e.*,

$$(s_1 - f(t, r_i, 0))(-1)^i < 0 \qquad \text{for all } t \in J,\ i \in \{1, \dots, n\},$$

and suppose that (50) *is fulfilled for all* $(t, x) \in J \times [s^*, r^*]$ *where* r^* *is given by* (51) *and* $s^* = r_1 - (b - a)\max\{|c_1|, c_2\}$, *then the assertions* (i)–(iv) *of Theorem* 4.5 *remain valid.*

Using one-sided growth conditions and Theorems 3.3., 3.4, 4.2, and 4.4, we get:

THEOREM 4.7. *Suppose* $k \in \{3, 4]\}$. *Let* $n \in \mathbf{N}$, $n > 2$ *be odd*, *and* $s_1, r_1, \dots, r_{n+1} \in \mathbf{R}$ *satisfy* (48) *and* (49). *Further*, *let* (21) *be fulfilled*. *Then there exists* $r^* \geq r_{n+1}$ *such that* (i)–(iv) *of Theorem* 4.5 *are valid.*

THEOREM 4.8. *Suppose* $k \in \{3, 4\}$. *Let* $n \in \mathbf{N}$, $n \geq 2$, *be even and let* $s_1, r_1, \dots, r_{n+1} \in \mathbf{R}$ *satisfy* (48) *and* (49). *Further*, *let* (22) *be fulfilled*. *Then there exists* $r^* \geq r_{n+1}$ *such that* (i)–(iv) *of Theorem* 4.5 *are valid.*

Similar results concerning the existence of two or three solutions of the periodic problem can be found in [3] and [13] also.

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